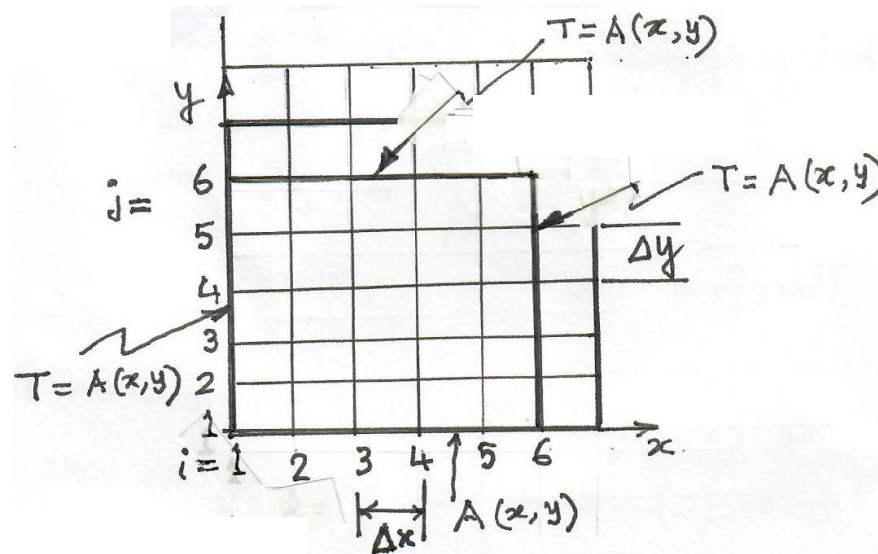


Heat and Mass transfer

Methods for solving Steady State Conduction Equations

Direct Methods for Solving Laplace Equation

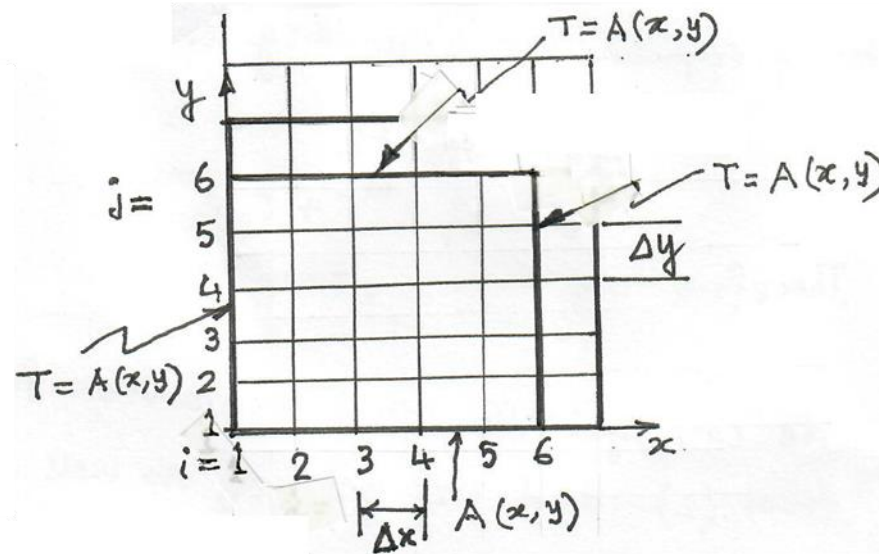
Direct methods are having little edge over iterative methods since round off errors do not exist and it involves predeterminable number operations.



- A square domain with Dirichlet boundary condition
- Let us assume a 6×6 grid and $\Delta x = \Delta y$

For each one of the points in the domain we write the difference equations in such a way that it becomes a system of 16 simultaneous linear algebraic equations for 16 unknown $T_{i,j}$ s.

Direct Methods for Solving Laplace Equation



The governing differential equation is a Laplace Equation for Temperature:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

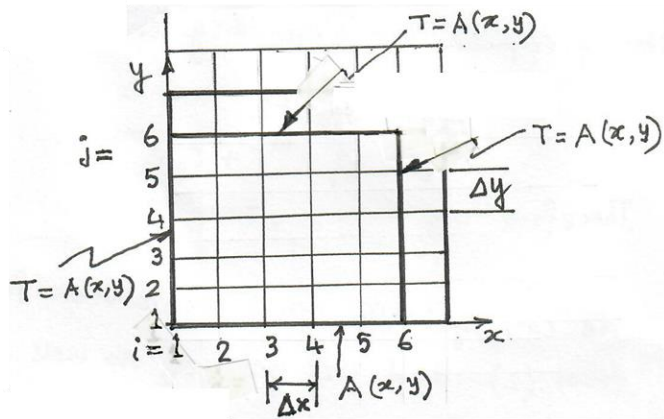
$$\text{or, } \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0$$

$$\text{for } \Delta x = \Delta y, \quad T_{i+1,j} + T_{i,j+1} + T_{i-1,j} + T_{i,j-1} - 4T_{i,j} = 0$$

Mathematically we want to formulate :

Mathematically we want to formulate:

$$\begin{array}{rcl} a_{11} T_1 + a_{12} T_2 + a_{13} T_3 + \dots + a_{1n} T_n & = & C_1 \\ a_{21} T_1 + a_{22} T_2 + a_{23} T_3 + \dots + a_{2n} T_n & = & C_2 \\ \vdots & & \vdots \\ a_{n1} T_1 + a_{n2} T_2 + a_{n3} T_3 + \dots + a_{nn} T_n & = & C_n \end{array}$$



Therefore, the nodal equations are:

$$\text{at } 2,2: A_{21}(x,y) + A_{12}(x,y) + T_{3,2} + T_{2,3} - 4T_{2,2} = 0$$

$$\text{or, } -4T_{2,2} + T_{3,2} + T_{2,3} = -A_{21}(x,y) - A_{12}(x,y)$$

$$\text{at } 3,2: A_{31}(x,y) + T_{2,2} + T_{3,3} - 4T_{3,2} + T_{4,2} = 0$$

$$\text{or, } T_{2,2} - 4T_{3,2} + T_{3,3} + T_{4,2} = -A_{31}(x,y)$$

$$\text{at } 4,2: A_{41}(x,y) + T_{3,2} + T_{4,3} + T_{5,2} - 4T_{4,2} = 0$$

$$\text{or, } T_{3,2} - 4T_{4,2} + T_{4,3} + T_{5,2} = -A_{41}(x,y)$$

$$\text{at } 5,2: A_{51}(x,y) + T_{4,2} - 4T_{5,2} + T_{5,3} + A_{62}(x,y) = 0$$

$$\text{or, } T_{4,2} - 4T_{5,2} + T_{5,3} = -A_{51}(x,y) - A_{62}(x,y)$$

and so on

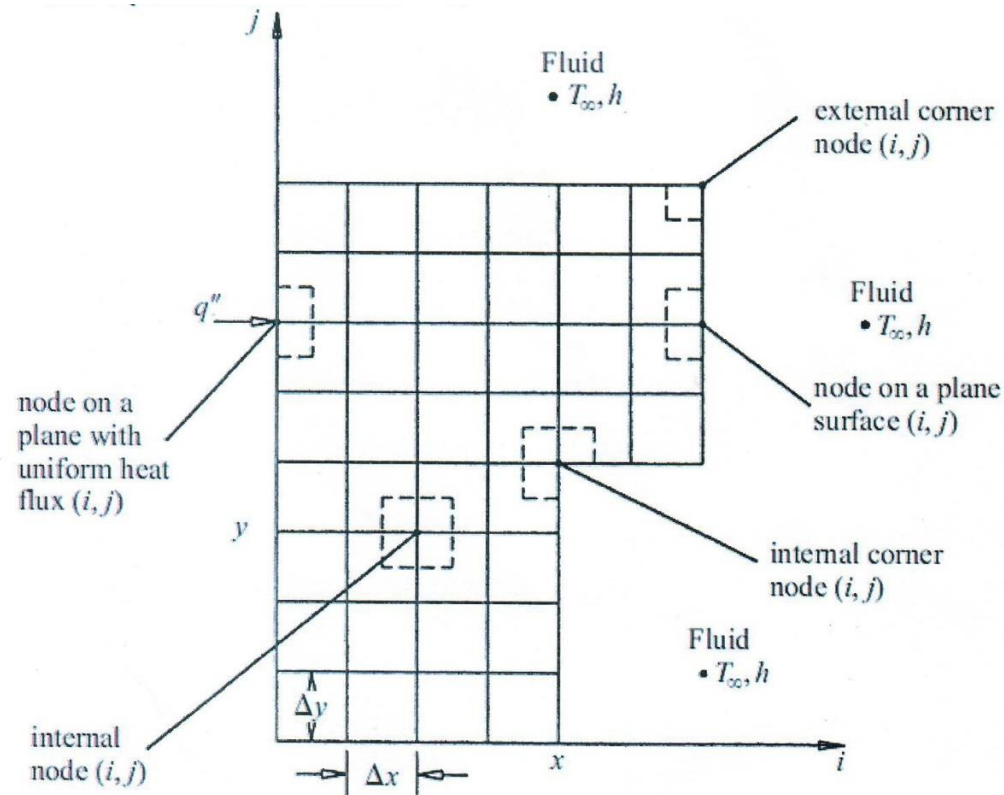
Finally we obtain

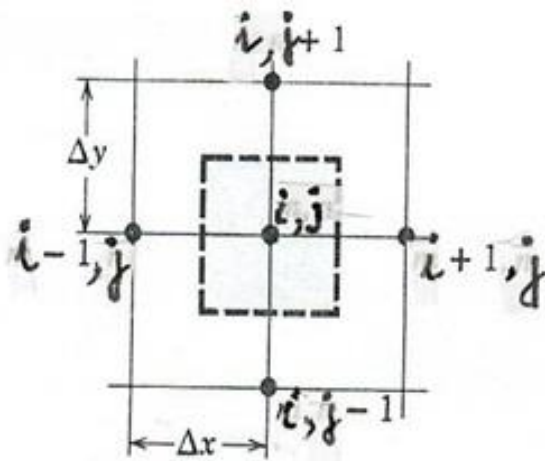
$$\begin{array}{c}
 \text{KNOWN} \\
 \left[\begin{array}{ccccccccc}
 c & & b & 0 & 0 & a & & 0 & 0 \\
 & b & & & & & a & & \\
 & 0 & & & & & & b & \\
 & 0 & & & & & & & a \\
 a & & b & & c & & & & \\
 & & & a & & & & & b \\
 & & & & & & & & \\
 0 & & & & & & & & \\
 0 & 0 & & a & 0 & 0 & b & & c
 \end{array} \right]
 \begin{bmatrix} T_{2,2} \\ T_{3,2} \\ T_{4,2} \\ T_{5,2} \\ T_{2,3} \\ T_{3,3} \\ T_{4,3} \\ UN \\ K \\ N \\ O \\ W \\ N \\ T_{5,5} \end{bmatrix}
 =
 \begin{bmatrix} d_{2,2} \\ d_{3,2} \\ d_{4,2} \\ d_{5,2} \\ d_{2,3} \\ d_{3,3} \\ d_{4,3} \\ K \\ N \\ O \\ W \\ N \\ d_{5,5} \end{bmatrix}
 \end{array}$$

which can be solved for $T_{i,j}$ s through an efficient matrix solver.

Finite-Difference Method

In the finite-difference approach, the conducting region is covered with a grid consisting of intersecting lines. The points of intersection are called nodes. For a rectangular region, the grid lines are drawn parallel to the boundaries. For simplicity, the spacings Δx and Δy are chosen so that $\Delta x = \Delta y$. Nodes are identified by double-subscript notation, i, j , where i and j count the grid lines along the x – and y – coordinate directions, respectively. The node i, j , which represents a particular subvolume, is presumed to be isothermal at the temperature, $T_{i,j}$



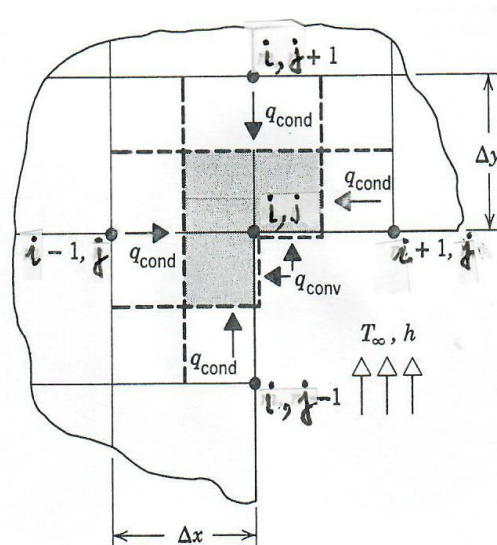


Derivations are based on a simple assumptions that each node represents centrain region. Its temperature is:

$$T_{i,j} = (T_{i+1,j} + T_{i,j+1} + T_{i-1,j} + T_{i,j-1}) / 4$$

The temperature $T_{i,j}$ at the nodal point may be viewed as the average temperature of the surrounding marked area.

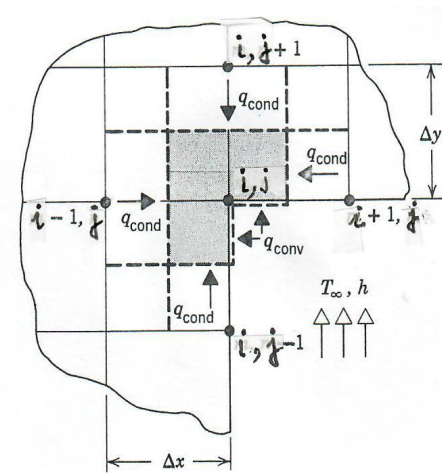
Now let us consider the node corresponding to the internal corner (i, j) . This node may be assumed to represent the three quarter shaded section and to exchange energy by convection with adjoining fluid at T_∞ . Conduction to the nodal region (i, j) will occur along four different routes from neighbouring nodes in the solid



$$q_{cond} =$$

$$k(\Delta y \cdot 1) \frac{(T_{i-1,j} - T_{i,j})}{\Delta x} + k(\Delta x \cdot 1) \frac{(T_{i,j+1} - T_{i,j})}{\Delta y} + k\left(\frac{\Delta y}{2} \cdot 1\right) \frac{(T_{i+1,j} - T_{i,j})}{\Delta x} + k\left(\frac{\Delta x}{2} \cdot 1\right) \frac{(T_{i,j-1} - T_{i,j})}{\Delta y} \quad - (a)$$

$$q_{conv} = h\left(\frac{\Delta x}{2} \cdot 1\right)(T_{\infty} - T_{i,j}) + h\left(\frac{\Delta y}{2} \cdot 1\right)(T_{\infty} - T_{i,j}) \quad - (b)$$



In absence of transient and generation effects, conservation of energy requires that $\Rightarrow (a) + (b) = 0$:

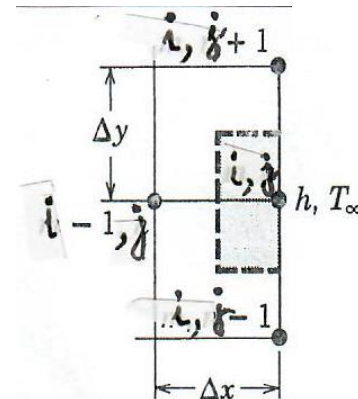
$$T_{i-1,j} + T_{i,j+1} + \frac{1}{2}(T_{i+1,j} + T_{i,j-1}) + \frac{h\Delta x}{k}T_{\infty} - \left(3 + \frac{h\Delta x}{k}\right)T_{i,j} = 0 \quad [\text{we have taken } \Delta x = \Delta y]$$

- $\frac{h\Delta x}{k}$ may be termed as the grid Biot number [Bi]. Let us quickly look at the situation,

when the node (i, j) is located at a plane surface with convection.

The local energy balance will produce

$$k(\Delta y \cdot 1) \frac{(T_{i-1,j} - T_{i,j})}{\Delta x} + k\left(\frac{\Delta x}{2} \cdot 1\right) \frac{(T_{i,j+1} - T_{i,j})}{\Delta y} + k\left(\frac{\Delta x}{2} \cdot 1\right) \frac{(T_{i,j-1} - T_{i,j})}{\Delta y} + h(\Delta y \cdot 1)(T_{\infty} - T_{i,j})$$

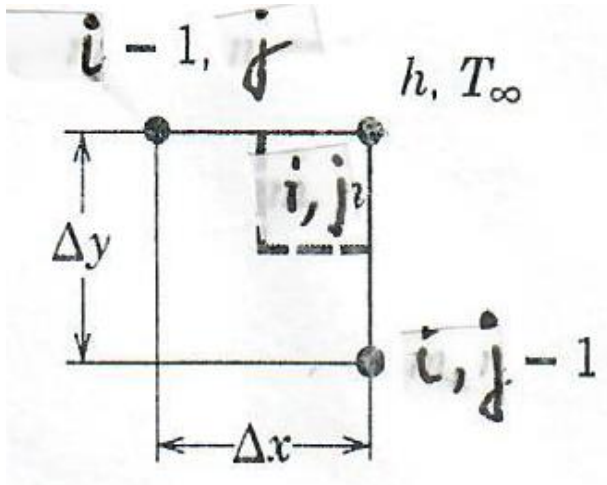


which finally leads to

$$\left(2T_{i-1,j} + T_{i,j+1} + T_{i,j-1}\right) + \frac{2h\Delta x}{k}T_{\infty} - 2\left(\frac{h\Delta x}{k} + 2\right)T_{i,j} = 0$$

If we take a medium of uniform volumetric heat generation, the resulting equation will be:

$$2T_{i-1,j} + T_{i,j+1} + T_{i,j-1} + \frac{2h\Delta x}{k}T_{\infty} - 2\left(\frac{h\Delta x}{k} + 2\right)T_{i,j} + \frac{q^*(\Delta x)^2}{k} = 0 \quad (\text{with } \Delta x = \Delta y)$$



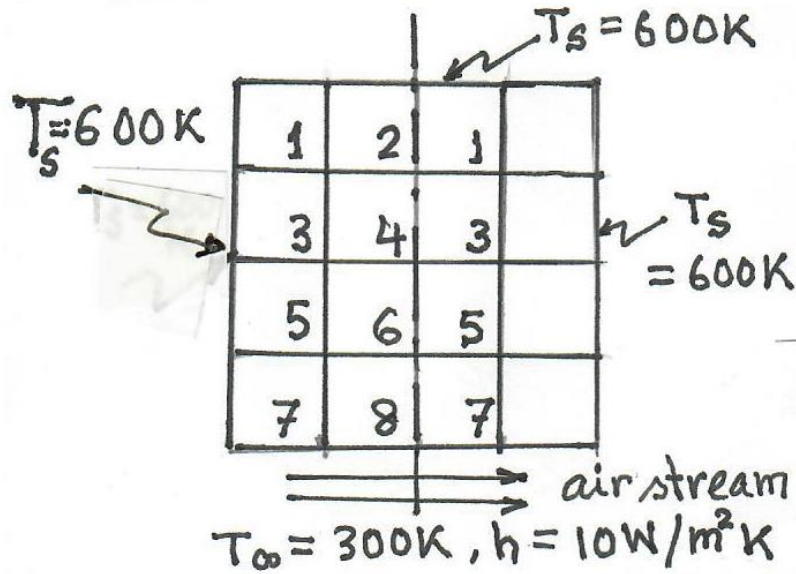
In a similar way, it can be shown that the finite difference equation for a node (i, j) at an external corner with convection is

$$\left(T_{i,j-1} + T_{i-1,j}\right) + \frac{2h\Delta x}{k}T_{\infty} - 2\left(\frac{h\Delta x}{k} + 1\right)T_{i,j} = 0$$

Now let us formulate the system of equations for a problem

Example

A 2-D plate of a given material is having dimension $1m \times 1m$. Three surfaces of the plate are maintained at $600K$. Remaining surface is exposed to air stream ($300K$). Using $\Delta x = \Delta y = 0.25m$, determine the temperature distribution. $h = 10W / m^2K$ and k of the material at $300K$ is $1W / mK$.



The grid consists of 12 nodal points.

Using symmetry about the midplane, the number of unknowns (T_i s) are reduced to 8

For node 1

$$T_2 + T_3 + 1200 - 4T_1 = 0$$

$$\text{or, } \underline{-4T_1 + T_2 + T_3 = -1200}$$

$$\text{at node 3: } T_1 + T_4 + T_5 + 600 - 4T_3 = 0$$

$$\text{or, } \underline{T_1 - 4T_3 + T_4 + T_5 = -600}$$

$$\text{at node 2: } 2T_1 + T_4 + 600 - 4T_2 = 0$$

$$\text{or, } \underline{2T_1 - 4T_2 + T_4 = -600}$$

$$\text{at node 5: } T_3 + T_6 + T_7 + 600 - 4T_5 = 0$$

$$\text{or, } \underline{T_3 - 4T_5 + T_6 + T_7 = -600}$$

$$\text{at node 4: } T_2 + 2T_3 + T_6 - 4T_4 = 0$$

$$\text{or, } \underline{T_2 + 2T_3 - 4T_4 + T_6 = 0}$$

at node 6:

$$T_4 + 2T_5 + T_8 - 4T_6 = 0$$

$$T_4 + 2T_5 - 4T_6 + T_8 = 0$$

For nodes 7 and 8, we have to make use of the relationship we have derived earlier. At node 7

$$2T_5 + T_8 + 600 + \frac{2 \times 10 \times 0.25}{1} \times 300 - 2 \left(\frac{10 \times 0.25}{1} + 2 \right) T_7 = 0$$

or, $2T_5 - 9T_7 + T_8 = -2100$

at node 8

$$2T_6 + 2T_7 + \frac{2 \times 10 \times 0.25}{1} \times 300 - 2 \left(\frac{10 \times 0.25}{1} + 2 \right) T_8 = 0$$

or, $2T_6 + 2T_7 - 9T_8 = -1500$

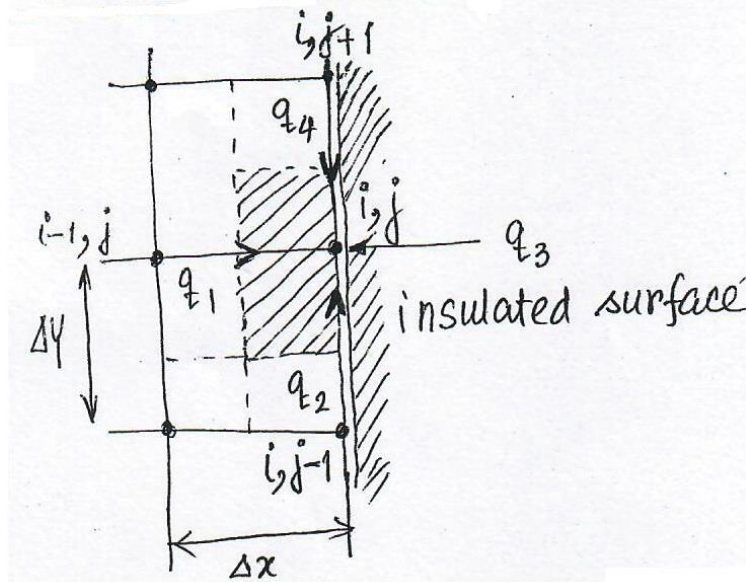
So, our system of equations will be:

$$\begin{bmatrix} -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & -9 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & -9 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{bmatrix} = \begin{bmatrix} -1200 \\ -600 \\ -600 \\ 0 \\ -600 \\ 0 \\ -2100 \\ -1500 \end{bmatrix}$$

Solution of this system of simultaneous equations will give the temperature distribution (steady state) on the plate.

We would like to discuss about a few more conditions which are important but they have not been discussed so far in the text.

Finite-difference equations at a nodal point on an insulated boundary



$$q_1 = k(\Delta y \cdot 1) \left(\frac{T_{i-1,j} - T_{i,j}}{\Delta x} \right)$$

$$q_2 = k \left(\frac{\Delta x}{2} \cdot 1 \right) \left(\frac{T_{i,j-1} - T_{i,j}}{\Delta y} \right)$$

$$q_3 = 0$$

$$q_4 = k \left(\frac{\Delta x}{2} \cdot 1 \right) \left(\frac{T_{i,j+1} - T_{i,j}}{\Delta y} \right)$$

Applying energy conservation requirement

$$k(\Delta y \cdot 1) \left(\frac{T_{i-1,j} - T_{i,j}}{\Delta x} \right) + k \cdot \frac{\Delta x}{2} \left(\frac{T_{i,j-1} - T_{i,j}}{\Delta y} \right) + k \cdot \frac{\Delta x}{2} \left(\frac{T_{i,j+1} - T_{i,j}}{\Delta y} \right) = 0$$

for $\Delta x = \Delta y$

$$2T_{i-1,j} + T_{i,j-1} + T_{i,j+1} - 4T_{i,j} = 0$$

If there is a volumetric heat generation at a rate of q^* , the modal equation will become

$$2T_{i-1,j} + T_{i,j-1} + T_{i,j+1} - 4T_{i,j} + \frac{q^* (\Delta x)^2}{k} = 0$$

Thank you